

RATIONAL REPRESENTATIONS AND PERMUTATION REPRESENTATIONS OF FINITE GROUPS

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ABSTRACT. We investigate the question which \mathbb{Q} -valued characters and characters of \mathbb{Q} -representations of finite groups are \mathbb{Z} -linear combinations of permutation characters. This question is known to reduce to that for quasi-elementary groups, and we give a solution in that case. As one of the applications, we exhibit a family of simple groups with rational representations whose smallest multiple that is a permutation representation can be arbitrarily large.

1. INTRODUCTION

Many rational invariants of a finite group G are encoded in the ring $\text{Char}_{\mathbb{Q}}(G)$ of rationally-valued characters, the ring $R_{\mathbb{Q}}(G)$ of rational representations, and the ring $\text{Perm}(G)$ of virtual permutation representations. All three have the same \mathbb{Z} -rank, and there are natural inclusions with finite cokernels

$$\text{Perm}(G) \longrightarrow R_{\mathbb{Q}}(G) \longrightarrow \text{Char}_{\mathbb{Q}}(G).$$

The quotient $\text{Char}_{\mathbb{Q}}(G)/R_{\mathbb{Q}}(G)$ is studied by the theory of Schur indices, and the purpose of this paper is to investigate the other two,

$$C(G) = \frac{R_{\mathbb{Q}}(G)}{\text{Perm}(G)} \quad \text{and} \quad \hat{C}(G) = \frac{\text{Char}_{\mathbb{Q}}(G)}{\text{Perm}(G)}.$$

They have exponent dividing $|G|$ by Artin's induction theorem, and Serre remarked that $C(G)$ need not be trivial ([14] Exc. 13.4). It is trivial for p -groups [6, 12, 13], and it is known for nilpotent groups [11] (see also §2), for Weyl groups of Lie groups [15, 9] and in other special cases [1, 7]. It follows from the general results of Dress, Kletzing, and Hambleton-Taylor-Williams [4, 5, 9, 8], that the study of $C(G)$ for a group G reduces, in principle, to that of its quasi-elementary subgroups, or of its 'basic' quasi-elementary subquotients. Specifically, for subgroups the statement is that of the two maps

$$\prod_{\substack{Q \leq G \\ \text{quasi-elem.}}} C(Q) \xrightarrow{\text{Ind}} C(G) \xrightarrow{\text{Res}} \prod_{\substack{Q \leq G \\ \text{quasi-elem.}}} C(Q),$$

the first one is surjective and the second one injective, and similarly for \hat{C} . This is also an immediate consequence of Solomon's induction theorem, see §3.

Our first observation is that the composite map allows us to describe $C(G)$ and $\hat{C}(G)$ explicitly, in a way that bypasses the representation theory of G — purely in terms of quasi-elementary subgroups and the 'Res Ind' maps between them; in fact, it is enough to consider maximal quasi-elementary subgroups, i.e. p -normalisers of cyclic subgroups of G . In §3 we give a simple formula for the Res Ind maps, and in §4 we prove one of the main results of the paper, which describes $C(Q)$ and $\hat{C}(Q)$ for a p -quasi-elementary group $Q = C \rtimes P$.

Its simplest formulation is:

Theorem 1.1 (=Theorem 4.6). *Let ρ be an irreducible rational representation of a p -quasi-elementary group $Q = C \rtimes P$. (So C is cyclic, P a p -group, and $p \nmid |C|$.) The order of ρ in $C(Q)$ is $\frac{\dim \hat{\psi} \dim \hat{\pi}}{\dim \rho}$, where $\hat{\psi}$ is the (unique) rational irreducible constituent of $\text{Res}_C \rho$ and $\hat{\pi}$ a rational irreducible constituent of $\text{Res}_P \rho$ of minimal dimension.*

Together with the aforementioned ‘Res Ind’ formula, it gives a way to compute $C(G)$ and $\hat{C}(G)$ efficiently in a given finite group G . Incidentally, it also gives an algorithm to find $\text{Perm}(G) \subset R_{\mathbb{Q}}(G)$ without computing the subgroup lattice, which is now implemented in Magma [2]. In §5 and §6 we illustrate applications of this approach to proving both triviality and non-triviality of $C(G)$, as we shall now describe.

In general, $C(G)$ remains somewhat mysterious, especially in non-soluble groups. Already Frobenius knew that $C(A_n)$ is trivial for all n . It was announced by Solomon in [15] that $C(\text{PSL}_2(\mathbb{F}_q))$ is trivial for all prime powers q . In §5 we explain how this, and the same statement for $\text{GL}_2(\mathbb{F}_q)$ and $\text{PGL}_2(\mathbb{F}_q)$, follow from the results of §3 and §4.

There is, to our knowledge, no example in the literature of a simple group with non-trivial $C(G)$. In §6 we show:

Theorem 1.2 (=Theorem 6.1 and Corollary 6.6). *The exponent of the 2-part of $C(G)$ is unbounded in the families $G = \text{PSL}_k(\mathbb{F}_p)$ and $G = \text{SL}_k(\mathbb{F}_p)$. Moreover, $\hat{C}(\text{PSL}_k(\mathbb{F}_p)) \neq \{1\}$ for all even $k \geq 4$ and all odd primes p .*

Notation. Throughout the paper, G denotes a finite group. We write

$$\begin{aligned} \text{Char}(G) &= \text{the character ring of } G, \\ \text{Char}_{\mathbb{Q}}(G) &= \text{the ring of } \mathbb{Q}\text{-valued characters,} \\ R_{\mathbb{Q}}(G) &= \text{the ring of characters of virtual } \mathbb{Q}G\text{-representations,} \\ \text{Perm}(G) &= \text{the ring of characters of virtual permutation representations,} \\ C(G) &= R_{\mathbb{Q}}(G)/\text{Perm}(G), \\ \hat{C}(G) &= \text{Char}_{\mathbb{Q}}(G)/\text{Perm}(G), \\ \mathbb{Q}(\chi) &= \text{the field of character values of a complex character } \chi, \\ m(\chi) &= \text{the Schur index of an irreducible complex character } \chi \text{ over } \mathbb{Q}(\chi). \end{aligned}$$

For a complex character χ of G , define its *trace* and, when χ is irreducible, its *rational hull* as

$$\begin{aligned} \text{Tr } \chi &= \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} \chi^{\sigma} && \in \text{Char}_{\mathbb{Q}}(G), \\ \hat{\chi} &= m(\chi) \text{Tr } \chi && \in R_{\mathbb{Q}}(G). \end{aligned}$$

If χ is irreducible, then $\text{Tr } \chi$ is a \mathbb{Q} -irreducible character and $\hat{\chi}$ is the character of an irreducible rational representation. We write

$$\begin{aligned} \text{Irr}(G) &= \text{the set of (complex) irreducible characters of } G, \\ \text{Irr}_{\mathbb{Q}}(G) &= \text{the set of } \mathbb{Q}\text{-irreducible characters of } G, \\ \mu(\alpha, \beta) &= \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} = \text{multiplicity of } \alpha \text{ in } \beta, \\ &\quad \text{used for characters } \alpha \in \text{Irr}_{\mathbb{Q}}(G), \beta \in \text{Char}_{\mathbb{Q}}(G), \text{ and} \\ &\quad \text{also for rational representations } \alpha, \beta \text{ with } \alpha \text{ irreducible.} \end{aligned}$$

We write $x \sim y$ for conjugate elements. A p -quasi-elementary group is one of the form $G = C \rtimes P$ with C cyclic, and P a p -group; throughout the paper we adopt the convention that $p \nmid |C|$.

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2. BASIC FACTS

Lemma 2.1. *An inclusion $N \triangleleft G$ induces injections $C(G/N) \hookrightarrow C(G)$, $\hat{C}(G/N) \hookrightarrow \hat{C}(G)$.*

Proof. Suppose $\bar{\rho}$ is a representation of G/N , which lifts to $\rho \in \text{Perm } G$. Write

$$\rho = \bigoplus \mathbb{C}[G/H_i]^{\oplus n_i}, \quad n_i \in \mathbb{Z}.$$

For a subgroup $H < G$ recall that $\mathbb{C}[G/H]^N \cong \mathbb{C}[G/NH]$, as G -representations (see e.g. [3], proof of Thm. 2.8(5)). Therefore,

$$\bar{\rho} = \rho^N = \bigoplus \mathbb{C}[G/NH_i]^{\oplus n_i} \in \text{Perm}(G/N),$$

as required. \square

Lemma 2.2. *Let ρ be an irreducible rational representation and $\tau \in \text{Irr } G$ its constituent, so $\text{Tr } \tau \in \text{Irr}_{\mathbb{Q}}(G)$ and $\rho = m(\tau) \text{Tr } \tau$. The order of $\text{Tr } \tau$ in $\hat{C}(G)$ is $m(\tau)$ times the order of ρ in $C(G)$.*

Proof. Clear from the definitions of $C(G)$ and $\hat{C}(G)$. \square

This allows us to immediately deduce results about $\hat{C}(G)$ from those about $C(G)$, and conversely.

Nilpotent groups. Some statements seem to have a cleaner formulation for $C(G)$ than for $\hat{C}(G)$, and for some it is the other way around. Let us briefly illustrate this with an example of nilpotent groups:

Theorem 2.3 (Rasmussen [11] Thm 5.2). *Let $G = G_2 \times G_{2'}$ be a nilpotent group, where G_2 is its Sylow 2-subgroup. Then $C(G)$ is trivial, unless $G_{2'} \neq \{1\}$ and there exists an irreducible character χ of G_2 with $m(\chi) = 2$ and such that one of the following holds:*

- (1) $\mathbb{Q}(\chi) \neq \mathbb{Q}$, or
- (2) $\mathbb{Q}(\chi) = \mathbb{Q}$ and there exists a prime divisor q of $|G|$ such that the order of 2 (mod q) is even.

The conditions turn out to be much simpler if one transforms this into a result about $\hat{C}(G)$. The following follows easily from [11, Thm. 4.2] and standard facts about Schur indices:

Theorem 2.4. *Let $\chi = \chi_2 \chi_{2'}$ be an irreducible character of a nilpotent group $G = G_2 \times G_{2'}$ as above. Then the order of $\text{Tr } \chi$ in $\hat{C}(G)$ is $m(\chi_2)$ (which is 1 or 2).*

Metabelian and supersolvable groups. The following theorem will be of central importance in what follows. It implies that knowing the order of every \mathbb{Q} -irreducible representation in $\hat{\mathbb{C}}(G)$ determines the structure of $\hat{\mathbb{C}}(G)$ completely when G is metabelian or supersolvable (e.g. nilpotent or quasi-elementary). It does not hold in arbitrary groups, as first noted by Berz [1]; the smallest counterexample is $G = C_3 \times \mathrm{SL}_2(\mathbb{F}_3)$.

Theorem 2.5 (Berz [1]). *If G is metabelian or supersolvable, then $\mathrm{Perm}(G) \subseteq R_{\mathbb{Q}}(G)$ is freely generated by $n_{\rho}\rho$, as ρ ranges over irreducible rational representations of G , and*

$$n_{\rho} = \gcd_{H \leq G} \mu(\rho, \mathbb{Q}[G/H]).$$

Lemma 2.6. *If $G = A \rtimes V$ with A abelian and V an elementary abelian p -group, then $\hat{\mathbb{C}}(G) = \{1\}$.*

Proof. By Theorem 2.5, it is enough to show that every complex irreducible character τ of G occurs exactly once in $\mathbb{C}[G/H]$ for a suitable $H < G$. This is clear when $\dim \tau = 1$. Otherwise $\tau = \mathrm{Ind}_{AU}^G \chi$, for some subgroup U of V and a 1-dimensional character χ of AU (see [14, Part II, §8.2]). Let H be a subgroup of V that is complementary to U , i.e. $HU = V$ and $H \cap U = \{1\}$. By Mackey's formula, we have

$$\langle \tau, \mathbb{C}[G/H] \rangle = \langle \chi, \mathrm{Res}_{AU} \mathrm{Ind}_H^G \mathbf{1} \rangle = \langle \chi, \mathrm{Ind}_{AU \cap H}^{AU} \mathbf{1} \rangle = \langle \chi, \mathbb{C}[AU] \rangle = 1. \quad \square$$

Recall that a p -quasi-elementary group $G = C \rtimes P$ is *basic* if the kernel K of $P \rightarrow \mathrm{Aut}(C)$ is trivial or isomorphic to D_8 or has normal p -rank one.

Proposition 2.7 ([7], Proposition 5.2). *Let $G = C \rtimes P$ be basic p -quasi-elementary. Let A_p be a maximal cyclic subgroup of $K = \ker(P \rightarrow \mathrm{Aut}(C))$ that is normal in P (it is all of K if K is cyclic, and has index 2 in K otherwise), let $A = CA_p$, and let χ be a faithful one-dimensional character of A . Then $\rho = \mathrm{Tr} \mathrm{Ind}_A^G \chi$ is a \mathbb{Q} -irreducible character, and*

$$\text{order of } \rho \text{ in } \hat{\mathbb{C}}(G) = \frac{|P|}{|A_p| \cdot \max_{\substack{H \leq P \\ H \cap A_p = 1}} |H|}.$$

3. $\hat{\mathbb{C}}(G)$ AS A MACKEY FUNCTOR

Let \mathcal{R} be a Mackey subfunctor of the character ring Mackey functor $\mathrm{Char}(G)$. This simply means that for any finite group G , $\mathcal{R}(G)$ is a subgroup of $\mathrm{Char}(G)$ such that if $H \leq G$ are finite groups, then

- for all $\rho \in \mathcal{R}(H)$, $\mathrm{Ind}_H^G \rho \in \mathcal{R}(G)$,
- for all $\tau \in \mathcal{R}(G)$, $\mathrm{Res}_H \tau \in \mathcal{R}(H)$,
- for all $\rho \in \mathcal{R}(H)$ and $g \in G$, $\rho^g \in \mathcal{R}(H^g)$.

Here are some examples:

- $R_K(G)$, the representation ring of G over a fixed subfield K of \mathbb{C} ,
- $\mathrm{Char}_K(G)$, the ring generated by K -valued characters, with fixed $K \subset \mathbb{C}$,
- $\mathrm{Perm}(G)$, the ring generated by permutation characters,
- the subgroup of $\mathrm{Char}(G)$ generated by characters of degree divisible by a fixed integer n .

If p is a prime number, write $\mathcal{R}(G)_p$ for $\mathcal{R}(G) \otimes \mathbb{Z}_p$, where \mathbb{Z}_p is the ring of p -adic integers.

Proposition 3.1. *Let G be a finite group, fix a prime number p , and let \mathcal{F}_p be a family of subgroups of G such that every p -quasi-elementary subgroup of G is conjugate to a subgroup of some $Q \in \mathcal{F}_p$. Then*

$$\prod_{Q \in \mathcal{F}_p} \text{Res}_Q : \mathcal{R}(G)_p \longrightarrow \prod_{Q \in \mathcal{F}_p} \mathcal{R}(Q)_p$$

is injective. Dually,

$$\sum_Q \text{Ind}_Q^G : \prod_{Q \in \mathcal{F}_p} \mathcal{R}(Q)_p \longrightarrow \mathcal{R}(G)_p$$

is surjective.

Proof. By Solomon's induction theorem, a prime-to- p multiple d of the trivial representation can be written as

$$d\mathbf{1}_G = \sum_i n_i \text{Ind}_{H_i}^G \mathbf{1}_{H_i}$$

for some p -quasi-elementary subgroups H_i and integers n_i . Because $\text{Ind}_{H_i}^G \mathbf{1}_{H_i} \cong \text{Ind}_{H_i}^G \mathbf{1}_{H_i}$, we may assume that each H_i is contained in some $Q_i \in \mathcal{F}_p$. Taking tensor products with any $\rho \in \mathcal{R}(G)$ yields

$$d\rho = \sum_i n_i \text{Ind}_{H_i}^G \text{Res}_{H_i} \rho.$$

If all $\text{Res}_{H_i} \rho$ were 0, then so would be $d\rho$, and therefore also ρ . This proves injectivity. Also, the equation shows that $d\rho \in \text{Im} \left(\sum_Q \text{Ind}_Q^G \mathcal{R}(Q) \right)$, which proves surjectivity, since d is invertible in \mathbb{Z}_p . \square

Corollary 3.2. *For $S, T \in \mathcal{F}_p$ write $\alpha_{S,T} = \text{Res}_T \text{Ind}_S^G : \hat{\mathbf{C}}(S) \longrightarrow \hat{\mathbf{C}}(T)$. Then*

$$\hat{\mathbf{C}}(G)_p \cong \text{Image} \left(\prod_T \sum_S \alpha_{S,T} : \prod_{S \in \mathcal{F}_p} \hat{\mathbf{C}}(S) \longrightarrow \prod_{T \in \mathcal{F}_p} \hat{\mathbf{C}}(T) \right).$$

In particular, $\hat{\mathbf{C}}(G)_p = 1$ if and only if for all pairs $S, T \in \mathcal{F}_p$ and all $\rho \in R_{\mathbb{Q}}(S)$ (equivalently, for those ρ whose class in $\mathbf{C}(S)$ is nontrivial), we have $\text{Res}_T^G \text{Ind}_S^G \rho \in \text{Perm}(T)$. The same also holds for $\mathbf{C}(G)$.

Proof. Apply Proposition 3.1 to \mathcal{R} being Perm , $R_{\mathbb{Q}}$, and $\text{Char}_{\mathbb{Q}}$. \square

Corollary 3.3. *Let \mathcal{F} be a family of subgroups of G such that every quasi-elementary subgroup is conjugate to a subgroup of some $Q \in \mathcal{F}$. Then*

$$\hat{\mathbf{C}}(G) \hookrightarrow \prod_{Q \in \mathcal{F}} \hat{\mathbf{C}}(Q)$$

via the (product of) restriction maps. Consequently, the kernel of the composition

$$R_{\mathbb{Q}}(G) \xrightarrow{\prod \text{Res}} \prod_{Q \in \mathcal{F}} R_{\mathbb{Q}}(Q) \longrightarrow \prod_{Q \in \mathcal{F}} \hat{\mathbf{C}}(Q)$$

is $\text{Perm}(G)$. Dually, the composition

$$\prod_{Q \in \mathcal{F}} R_{\mathbb{Q}}(Q) \xrightarrow{\text{Ind}} R_{\mathbb{Q}}(G) \rightarrow \hat{\mathbf{C}}(G)$$

is onto. The same holds with $R_{\mathbb{Q}}$ replaced by $\text{Char}_{\mathbb{Q}}$ and \hat{C} by C .

Remark 3.4. The theorem and the two corollaries give a very efficient way of computing $\hat{C}(G)_p, \hat{C}(G), C(G)_p, C(G)$ and of finding $\text{Perm}(G)$ as a subring of $R_{\mathbb{Q}}(G) \leq \text{Char}_{\mathbb{Q}}(G)$, without computing the full lattice of subgroups of G .

Remark 3.5. One possible family \mathcal{F}_p is the set of maximal p -quasi-elementary subgroups of G . These are of the form

$$Q = C \rtimes \text{Syl}_p(N_G(C)),$$

where C is cyclic of order prime to p . Possible families \mathcal{F} in Corollary 3.3 are $\mathcal{F} = \bigcup_p \mathcal{F}_p$, as p ranges over prime divisors of $|G|$, or alternatively $\mathcal{F} = \{N_G(C)\}$ as C ranges over (representatives of conjugacy classes of) cyclic subgroups of G .

Notation 3.6. For the remainder of this section we use the following notation:

$$\begin{aligned} CC(G) &= \text{the set of conjugacy classes of } G, \\ CC_{\text{cyc}}(G) &= \text{the set of conjugacy classes of cyclic subgroups of } G, \\ [x] &= \text{the conjugacy class of } x, \text{ when } x \text{ is either an element of } G \\ &\quad \text{or a cyclic subgroup,} \\ \text{Tr}^* \chi &= \text{the normalised trace } \text{Tr}^* \chi = \frac{1}{|\mathbb{Q}(\chi):\mathbb{Q}|} \text{Tr } \chi \text{ of a character } \chi, \\ \tau(D) &= \tau(y), \text{ where } D \leq G \text{ is a cyclic subgroup, } y \text{ is any generator} \\ &\quad \text{of } D, \text{ and } \tau \in \text{Char}_{\mathbb{Q}}(G) \otimes \mathbb{Q}. \text{ The rationality of } \tau \text{ ensures} \\ &\quad \text{that } \tau(y) \text{ only depends on } D \text{ and not on the generator } y. \end{aligned}$$

Note in particular, that for any character χ of G and any cyclic subgroup D of G , $\text{Tr}^* \chi(D)$ is the average value of χ on the generators of D .

Lemma 3.7. Let H_1, H_2 be two subgroups of G . Let τ_i be a character of H_i , $i = 1, 2$, and assume that τ_1 is \mathbb{Q} -valued. Then

$$\langle \text{Ind}_{H_1}^G \tau_1, \text{Ind}_{H_2}^G \tau_2 \rangle = \frac{1}{|H_1||H_2|} \sum_{[C] \in CC_{\text{cyc}}(G)} |N_G(C)| \phi(|C|) \cdot \sum_{\substack{D_1 \leq H_1 \\ D_1 \sim C}} \tau_1(D_1) \cdot \sum_{\substack{D_2 \leq H_2 \\ D_2 \sim C}} \text{Tr}^* \tau_2(D_2).$$

Proof. First, note that by definition of inner products and of induced class functions,

$$\langle \text{Ind}_{H_1}^G \tau_1, \text{Ind}_{H_2}^G \tau_2 \rangle = \frac{1}{|H_1||H_2|} \sum_{[x] \in CC(G)} |Z_G(x)| \overline{\left(\sum_{y \in [x] \cap H_1} \tau_1(y) \right)} \left(\sum_{y \in [x] \cap H_2} \tau_2(y) \right).$$

The idea of the proof is to partition the set of conjugacy classes of elements of G according to conjugacy classes of cyclic subgroups they generate, and to use the fact that for a rational character τ , $\tau(x) = \tau(x')$ whenever x and x' generate conjugate cyclic subgroups. We get

$$\begin{aligned} &\langle \text{Ind}_{H_1}^G \tau_1, \text{Ind}_{H_2}^G \tau_2 \rangle \\ &= \frac{1}{|H_1||H_2|} \sum_{[x] \in CC(G)} |Z_G(x)| \overline{\left(\sum_{y \in [x] \cap H_1} \tau_1(y) \right)} \left(\sum_{y \in [x] \cap H_2} \tau_2(y) \right) \\ &= \frac{1}{|H_1||H_2|} \sum_{[C] \in CC_{\text{cyc}}(G)} f(C), \end{aligned}$$

where

$$\begin{aligned}
f(C) &= |Z_G(C)| \cdot \sum_{\substack{[x] \in CC(G) \\ \langle x \rangle = C}} \left(\sum_{y \in [x] \cap H_1} \tau_1(y) \right) \left(\sum_{y \in [x] \cap H_2} \tau_2(y) \right) \\
&= |Z_G(C)| \cdot \#\{k : x \sim x^k\} \cdot \sum_{\substack{D_1 \leq H_1 \\ D_1 \sim C}} \tau_1(D_1) \cdot \sum_{\substack{[x] \in CC(G) \\ [x] \sim C}} \sum_{y \in [x] \cap H_2} \tau_2(y) \\
&= |N_G(C)| \cdot \sum_{\substack{D_1 \leq H_1 \\ D_1 \sim C}} \tau_1(D_1) \cdot \sum_{\substack{D_2 \leq H_2 \\ D_2 \sim C}} \sum_{\substack{\text{generators} \\ y \text{ of } D_2}} \tau_2(y) \\
&= |N_G(C)| \cdot \sum_{\substack{D_1 \leq H_1 \\ D_1 \sim C}} \tau_1(D_1) \cdot \sum_{\substack{D_2 \leq H_2 \\ D_2 \sim C}} \phi(|C|) \text{Tr}_{\mathbb{Q}/\mathbb{Q}}^* \tau_2(y),
\end{aligned}$$

as claimed. \square

Corollary 3.8. *Suppose $H_1 < Q_1 < G$, $H_2 < Q_2 < G$, and let χ_i be irreducible characters of H_i . Set $\tau_i = \text{Ind}_{H_i}^{Q_i} \chi_i$, and $\rho_i = \text{Tr } \tau_i$. Assume that τ_2 is irreducible. Then*

$$\begin{aligned}
\mu(\rho_2, \text{Res}_{Q_2} \text{Ind}_{Q_1}^G \rho_1) &= \frac{[\mathbb{Q}(\tau_1) : \mathbb{Q}]}{|H_1| \cdot |H_2|} \sum_{[C] \in CC_{\text{cyc}}(G)} |N_G(C)| \phi(|C|) \cdot \\
&\quad \cdot \sum_{\substack{D_1 \leq H_1 \\ D_1 \sim C}} \text{Tr}^* \chi_1(D_1) \cdot \sum_{\substack{D_2 \leq H_2 \\ D_2 \sim C}} \text{Tr}^* \chi_2(D_2).
\end{aligned}$$

Proof.

$$\begin{aligned}
\mu(\rho_2, \text{Res}_{Q_2} \text{Ind}_{Q_1}^G \rho_1) &= \langle \tau_2, \text{Res}_{Q_2} \text{Ind}_{Q_1}^G (\sum \tau_1^\sigma) \rangle \\
&= \langle \text{Ind}_{H_2}^G \chi_2, \text{Ind}_{H_1}^G (\sum_{\sigma \in \text{Gal}(\mathbb{Q}(\tau_1)/\mathbb{Q})} (\chi_1)^\sigma) \rangle \\
&= \frac{1}{[\mathbb{Q}(\chi_1) : \mathbb{Q}(\tau_1)]} \langle \text{Ind}_{H_2}^G \chi_2, \text{Ind}_{H_1}^G (\text{Tr } \chi_1) \rangle \\
&= \frac{1}{[\mathbb{Q}(\chi_1) : \mathbb{Q}(\tau_1)] |H_1| \cdot |H_2|} \sum_{[C] \in CC_{\text{cyc}}(G)} |N_G(C)| \phi(|C|) \cdot \\
&\quad \cdot \sum_{\substack{D_1 \leq H_1 \\ D_1 \sim C}} \text{Tr } \chi_1(D_1) \cdot \sum_{\substack{D_2 \leq H_2 \\ D_2 \sim C}} \text{Tr}^* \chi_2(D_2) \\
&= \frac{[\mathbb{Q}(\tau_1) : \mathbb{Q}]}{|H_1| \cdot |H_2|} \sum_{[C] \in CC_{\text{cyc}}(G)} |N_G(C)| \phi(|C|) \cdot \\
&\quad \cdot \sum_{\substack{D_1 \leq H_1 \\ D_1 \sim C}} \text{Tr}^* \chi_1(D_1) \cdot \sum_{\substack{D_2 \leq H_2 \\ D_2 \sim C}} \text{Tr}^* \chi_2(D_2).
\end{aligned}$$

\square

Lemma 3.9. *If C is a cyclic group, and χ is a 1-dimensional character of C , then $(\text{Tr}^* \chi)(C) = \mu(\text{ord}(\chi)) / \phi(\text{ord}(\chi))$, where μ is the Moebius mu function, and $\text{ord}(\chi)$ is the smallest natural number n such that $\chi^n = \mathbf{1}$.*

Proof. It is enough to prove the lemma for faithful characters χ , since we may, without loss of generality, replace C by $C / \ker \chi$. Let g be a generator of C . Then

$$(\text{Tr}^* \chi)(C) = \frac{1}{[\mathbb{Q}(\chi) : \mathbb{Q}]} \text{Tr } \chi(g) = \frac{1}{\phi(\text{ord}(\chi))} \text{Tr } \chi(g).$$

If $|C| = n$, then $\chi(g)$ is a primitive n -th root of unity, and the fact that its trace is $\mu(n)$ is classical. \square

Corollary 3.10. *Let G be a group and p^r a prime power. Then $\hat{C}(G)$ has an element of order p^r if and only if there exist two p -quasi-elementary subgroups Q_1, Q_2 of G , irreducible monomial characters $\tau_i = \text{Ind}_{H_i}^{Q_i} \chi_i$ of Q_i , and an integer k , such that*

- the rational character $\text{Tr } \tau_2$ has order divisible by p^{k+r} in $\hat{C}(Q_2)$, and
- the rational number

$$\frac{[\mathbb{Q}(\tau_1) : \mathbb{Q}]}{|H_1||H_2|} \cdot \sum_{\substack{[U] \in CC_{\text{cyc}}(G) \\ D_1 \leq H_1 \\ D_1 \sim U}} |N_G(U)| \phi(|U|) \cdot \sum_{\substack{D_1 \leq H_1 \\ D_1 \sim U}} \frac{\mu([D_1 : D_1 \cap \ker \chi_1])}{\phi([D_1 : D_1 \cap \ker \chi_1])} \cdot \sum_{\substack{D_2 \leq H_2 \\ D_2 \sim U}} \frac{\mu([D_2 : D_2 \cap \ker \chi_2])}{\phi([D_2 : D_2 \cap \ker \chi_2])}$$

has p -adic valuation at most k .

In this case, $\text{Ind}_{Q_1}^G \text{Tr } \tau_1$ has order divisible by p^r in $\hat{C}(G)$.

Remark 3.11.

- Note that it is enough to take the last two sums in the above formula only over those D_i for which $D_i \cap \ker \chi_i$ has square-free index in D_i , since for the others $\mu(\text{ord}(\text{Res}_{D_i} \chi_i)) = 0$. For example if χ_i are faithful, then the outer sum may be taken over U of square free order.
- If, say, H_1 is cyclic, the sum $\sum_{\substack{D_1 \leq H_1 \\ D_1 \sim U}}$ has at most one term for every U .
- If Q_1, Q_2 are basic and H_1, H_2 are cyclic, then Proposition 2.7 gives a simple expression for the order of $\text{Tr } \tau_2$ in $\hat{C}(Q_2)$.

Proof of Corollary 3.10. By Corollary 3.2, $\hat{C}(G)_p$ has an element of order p^r if and only if there exist p -quasi-elementary subgroups Q_1, Q_2 , and characters $\rho_i \in \text{Irr}_{\mathbb{Q}}(Q_i)$, such that ρ_2 has order p^{k+r} in $\hat{C}(Q_2)$ for some k , and $\mu(\rho_2, \text{Res}_{Q_2} \text{Ind}_{Q_1}^G \rho_1)$ has p -adic valuation at most k . Quasi-elementary groups are M-groups, so if τ_i is a complex irreducible constituent of ρ_i , then there exist subgroups $H_i \leq Q_i$ such that $\tau_i = \text{Ind}_{H_i}^{Q_i} \chi_i$ for 1-dimensional characters $\chi_i \in \text{Irr}(H_i)$. The result therefore follows from Corollary 3.8 in combination with Lemma 3.9. \square

4. QUASI-ELEMENTARY GROUPS

The aim of this section is to provide several formulae of theoretical and algorithmic interest for the orders of characters in $\hat{C}(G)$ and $C(G)$ when G is quasi-elementary. Let $G = C \rtimes P$ with P a p -group and C cyclic of order coprime to p ; we identify P with a Sylow subgroup of G .

Lemma 4.1. *Let N be a normal subgroup of a finite group G , let η be an irreducible character of N , and let θ be a complex irreducible constituent of $\text{Ind}_N^G \eta$. Write $\mathcal{G}_\eta = \text{Gal}(\mathbb{Q}(\eta)/\mathbb{Q})$, and similarly for \mathcal{G}_θ . Then*

$$\frac{[\mathbb{Q}(\eta) : \mathbb{Q}]}{[\mathbb{Q}(\theta) : \mathbb{Q}]} = \frac{\#\{\gamma \in \mathcal{G}_\eta \mid \langle \eta^\gamma, \text{Res}_N \theta \rangle \neq 0\}}{\#\{\gamma \in \mathcal{G}_\theta \mid \langle \text{Ind}_N^G \eta, \theta^\gamma \rangle \neq 0\}}.$$

In particular, if $\text{Ind}_N^G \eta$ is irreducible, then

$$\frac{[\mathbb{Q}(\eta) : \mathbb{Q}]}{[\mathbb{Q}(\theta) : \mathbb{Q}]} = \#\{\gamma \in \mathcal{G}_\eta \mid \langle \eta^\gamma, \text{Res}_N \theta \rangle \neq 0\}.$$

Proof. The G -action on the characters of N commutes with the Galois action. Every Galois conjugate of θ is a constituent of $\text{Ind}_N^G \eta^\gamma$ for some $\gamma \in \mathcal{G}_\eta$, and moreover the number of distinct Galois conjugates of θ in η^γ is independent of γ .

Also, the number of Galois conjugates of η in $\text{Res}_N \theta^\gamma$ is independent of $\gamma \in \mathcal{G}_\theta$. So an inclusion-exclusion count gives

$$\#\mathcal{G}_\theta = \#\mathcal{G}_\eta \cdot \frac{\#\{\gamma \in \mathcal{G}_\theta \mid \langle \text{Ind}_N^G \eta, \theta^\gamma \rangle \neq 0\}}{\#\{\gamma \in \mathcal{G}_\eta \mid \langle \eta^\gamma, \text{Res}_N \theta \rangle \neq 0\}}.$$

□

Lemma 4.2. *Let η be an irreducible complex representation of G , with rational hull $\hat{\eta}$. Then*

$$\dim \hat{\eta} = \dim \eta \cdot m(\eta) \cdot [\mathbb{Q}(\eta) : \mathbb{Q}].$$

Proof. The rational hull of η is given by

$$\hat{\eta} = m(\eta) \sum_{\gamma \in \text{Gal}(\mathbb{Q}(\eta)/\mathbb{Q})} \eta^\gamma,$$

whence the claim follows. □

Theorem 4.3. *Let $G = C \rtimes X$ with C cyclic of order coprime to $|X|$. Let τ be a complex irreducible character of G with rational hull $\rho = \hat{\tau}$, let π be a complex irreducible constituent of $\text{Res}_X \tau$ with rational hull $\hat{\pi}$, ψ an irreducible constituent of $\text{Res}_C \tau$ with rational hull $\hat{\psi}$, K_ψ the stabiliser of ψ under the X -action on $\text{Irr}(C)$, and let ξ be a complex irreducible constituent of $\text{Res}_{K_\psi} \pi$. Then*

$$\begin{aligned} \mu(\rho, \text{Ind}_X^G \hat{\pi}) &= \\ &= \frac{m(\pi)}{m(\tau)} \langle \xi, \text{Res}_{K_\psi} \pi \rangle \cdot \#\{\text{Galois conjugates } \pi' \text{ of } \pi \mid \langle \text{Res}_{K_\psi} \pi', \xi \rangle \neq 0\} \\ &= \frac{\dim \hat{\psi} \dim \hat{\pi}}{\dim \rho}. \end{aligned}$$

Proof. We may assume that $\rho|_C$ is faithful, otherwise we prove the result in the quotient $G/(\ker \rho \cap C)$. So $K = K_\psi$ is assumed to be the kernel of the X -action on C . Recall that ψ denotes a complex constituent of $\tau|_C$. In particular, $\tau = \text{Ind}_{CK}^G \psi \xi$, as explained in [14, Part II, §8.2]. We have

$$\begin{aligned} \rho &= m(\tau) \sum_{\gamma \in \text{Gal}(\mathbb{Q}(\tau)/\mathbb{Q})} \tau^\gamma; & \dim \rho &= m(\tau) [\mathbb{Q}(\tau) : \mathbb{Q}] \dim \tau, \\ \hat{\pi} &= m(\pi) \sum_{\gamma \in \text{Gal}(\mathbb{Q}(\pi)/\mathbb{Q})} \pi^\gamma; & \dim \hat{\pi} &= m(\pi) [\mathbb{Q}(\pi) : \mathbb{Q}] \dim \pi. \end{aligned}$$

Thus

$$\begin{aligned} \mu(\rho, \text{Ind}_X^G \hat{\pi}) &= \frac{1}{m(\tau)} \langle \tau, \text{Ind}_X^G \hat{\pi} \rangle = \frac{1}{m(\tau)} \langle \text{Ind}_{CK}^G \psi \xi, \text{Ind}_X^G \hat{\pi} \rangle \\ &= \frac{1}{m(\tau)} \langle \text{Res}_X \text{Ind}_{CK}^G \psi \xi, \hat{\pi} \rangle = \frac{1}{m(\tau)} \langle \text{Ind}_K^X \xi, \hat{\pi} \rangle = \frac{1}{m(\tau)} \langle \xi, \text{Res}_K \hat{\pi} \rangle, \end{aligned}$$

where the last line follows from Mackey's formula, noting that $CK \backslash G/X$ consists of one double coset, and that $CK \cap X = K$.

Next, X acts on the representations of K by conjugation, and there is a Clifford theory decomposition

$$(4.4) \quad \text{Res}_K \pi = e \sum_{g \in X / \text{Stab}_X \xi} \xi^g.$$

Recall that the constituents of $\hat{\pi}$ are Galois conjugates of π , and we select those whose restriction to K contains ξ :

$$\Omega = \{\gamma \in \text{Gal}(\mathbb{Q}(\pi) : \mathbb{Q}) \mid \langle \text{Res}_K \pi^\gamma, \xi \rangle \neq 0\}.$$

The inner product $\langle \text{Res}_K \pi^\gamma, \xi \rangle = \langle \text{Res}_K \pi, \xi^{\gamma^{-1}} \rangle$ is the same (and equals e) for every $\gamma \in \Omega$, since $\xi^{\gamma^{-1}}$ is irreducible and so must be one of ξ^g in (4.4). So we have

$$\frac{1}{m(\tau)} \langle \xi, \text{Res}_K \hat{\pi} \rangle = \frac{m(\pi)}{m(\tau)} |\Omega| \langle \xi, \text{Res}_K \pi \rangle,$$

which proves the first equality.

It remains to show that

$$(4.5) \quad \frac{m(\pi)}{m(\tau)} |\Omega| \langle \xi, \text{Res}_K \pi \rangle = \frac{\dim \hat{\psi} \dim \hat{\pi}}{\dim \rho}.$$

By comparing dimensions in (4.4), and since $\tau = \text{Ind}_{CK}^G \psi \xi$, we see that

$$\langle \xi, \text{Res}_K \pi \rangle = e = \frac{\dim \pi}{[X : \text{Stab}_X \xi] \dim \xi} = \frac{[X : K] \dim \pi}{[X : \text{Stab}_X \xi] \dim \tau} = \frac{[\text{Stab}_X \xi : K] \dim \pi}{\dim \tau},$$

so

$$\mu(\rho, \text{Ind}^G \hat{\pi}) = \frac{m(\pi)}{m(\tau)} |\Omega| \langle \xi, \text{Res}_K \pi \rangle = |\Omega| \cdot [\text{Stab}_X \xi : K] \frac{m(\pi) \dim \pi}{m(\tau) \dim \tau}.$$

Consider the two groups

$$\begin{aligned} H_1 &= \{\gamma \in \text{Gal}(\mathbb{Q}(\psi\xi)/\mathbb{Q}) \mid \langle (\psi\xi)^\gamma, \text{Res}_{CK} \tau \rangle \neq 0\}, \\ H_2 &= \{\gamma \in \text{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}) \mid \langle \xi^\gamma, \text{Res}_K \pi \rangle \neq 0\}. \end{aligned}$$

There is a natural projection $H_1 \twoheadrightarrow H_2$ given by the restriction of Galois action to $\mathbb{Q}(\xi)$, whose kernel consists of precisely those elements of $\text{Gal}(\mathbb{Q}(\psi\xi)/\mathbb{Q})$ that act trivially on ξ , and through the action of some $g \in X$ on ψ (this last condition is equivalent to the Galois element being in H_1). Thus, the kernel is isomorphic to the subgroup of G/CK that acts trivially on ξ , i.e. to $\text{Stab}_X \xi/K$. We deduce that

$$\mu(\rho, \text{Ind}^G \hat{\pi}) = |\Omega| \frac{|H_1|}{|H_2|} \frac{m(\pi) \dim \pi}{m(\tau) \dim \tau}.$$

Now, by applying Lemma 4.1 first to $CK \triangleleft G$ with $\theta = \tau$, $\eta = \psi\xi$, and then to $K \triangleleft X$ with $\theta = \pi$, $\eta = \xi$, we find that

$$|H_1| = \frac{[\mathbb{Q}(\xi) : \mathbb{Q}][\mathbb{Q}(\psi) : \mathbb{Q}]}{[\mathbb{Q}(\tau) : \mathbb{Q}]} \quad \text{and} \quad |H_2| = |\Omega| \frac{[\mathbb{Q}(\xi) : \mathbb{Q}]}{[\mathbb{Q}(\pi) : \mathbb{Q}]},$$

so that

$$\begin{aligned} \mu(\rho, \text{Ind}^G \hat{\pi}) &= |\Omega| \cdot \frac{|H_1|}{|H_2|} \frac{m(\pi) \dim \pi}{m(\tau) \dim \tau} \\ &= |\Omega| \cdot \frac{[\mathbb{Q}(\xi) : \mathbb{Q}] \cdot [\mathbb{Q}(\psi) : \mathbb{Q}]/[\mathbb{Q}(\tau) : \mathbb{Q}]}{|\Omega| [\mathbb{Q}(\xi) : \mathbb{Q}]/[\mathbb{Q}(\pi) : \mathbb{Q}]} \cdot \frac{m(\pi) \dim \pi}{m(\tau) \dim \tau} \\ &= \frac{[\mathbb{Q}(\psi) : \mathbb{Q}] \cdot [\mathbb{Q}(\pi) : \mathbb{Q}] m(\pi) \dim \pi}{[\mathbb{Q}(\tau) : \mathbb{Q}] m(\tau) \dim \tau} = \frac{\dim \hat{\psi} \dim \hat{\pi}}{\dim \rho}, \end{aligned}$$

where the last equality follows from Lemma 4.2. \square

Theorem 4.6. *Let $G = C \rtimes P$ be p -quasi-elementary, let ρ be an irreducible rational representation of G . Let ψ be a complex irreducible constituent of $\text{Res}_C \rho$ with rational hull $\hat{\psi}$, and let $\hat{\pi}$ be a rational irreducible constituent of $\text{Res}_P \rho$ of minimal dimension. Denote by π a complex irreducible constituent of $\hat{\pi}$, by ξ a complex irreducible constituent of $\pi|_{K_\psi}$, where $K_\psi \leq P$ is the stabiliser in P of ψ , and by τ a complex irreducible constituent of ρ such that $\text{Res}_P \tau$ contains π . Then*

$$\begin{aligned} \text{order of } \rho \text{ in } \mathbb{C}(G) &= \mu(\rho, \text{Ind}_P^G \hat{\pi}) = \frac{\dim \hat{\psi} \dim \hat{\pi}}{\dim \rho} \\ &= \frac{m(\pi)}{m(\tau)} \langle \xi, \text{Res}_{K_\psi} \pi \rangle \cdot \#\{\text{Galois conjugates } \pi' \text{ of } \pi \mid \xi \subset \text{Res}_{K_\psi} \pi'\}. \end{aligned}$$

Proof. We may assume that $\rho|_C$ is faithful, otherwise we prove the result in the quotient $G/(\ker \rho \cap C)$ (see Lemma 2.1). Thus, $K = K_\psi$ is assumed to be the kernel of the P -action on C . Under this assumption, if $H \leq G$ intersects C non-trivially, then

$$\langle \rho, \mathbb{C}[G/H] \rangle_G = \langle \text{Res}_H \rho, \mathbf{1} \rangle_H = 0.$$

Write o for the order of ρ in $\mathbb{C}(G)$. By Theorem 2.5, we have

$$\begin{aligned} o \cdot \langle \rho, \rho \rangle &= \gcd_{H \leq G} \langle \rho, \mathbb{C}[G/H] \rangle_G = \gcd_{H \leq P} \langle \rho, \mathbb{C}[G/H] \rangle_G \\ &= \gcd_{H \leq P} \langle \rho|_P, \mathbb{C}[P/H] \rangle_P. \end{aligned}$$

Because $\mathbb{C}(P) = 1$ by the Ritter-Segal theorem [12, 13], we can replace the permutation representations $\mathbb{C}[P/H]$ by all rational representations of P in the last term. This is clearly the same as just taking the rational irreducible constituents $\hat{\pi}_1, \dots, \hat{\pi}_k$ of $\rho|_P$, so

$$(4.7) \quad o = \frac{1}{\langle \rho, \rho \rangle} \gcd_j \langle \rho|_P, \hat{\pi}_j \rangle = \gcd_j \frac{\langle \rho, \text{Ind}_P^G \hat{\pi}_j \rangle}{\langle \rho, \rho \rangle} = \gcd_j \mu(\rho, \text{Ind}_P^G \hat{\pi}_j).$$

The theorem will therefore follow from Theorem 4.3, once we show that the gcd may be replaced by the term corresponding to any $\hat{\pi}$ of minimal dimension. Now, by Theorem 4.3 and by Lemma 4.2,

$$\mu(\rho, \text{Ind}_P^G \hat{\pi}_j) = \frac{\dim \hat{\psi} \dim \hat{\pi}_j}{\dim \rho} = \frac{\dim \hat{\psi} m(\pi_j) \dim \pi_j [\mathbb{Q}(\pi_j) : \mathbb{Q}]}{\dim \rho},$$

where π_j is a complex irreducible constituent of $\hat{\pi}_j$. We argue as in [17, §2]: if $p = 2$, then all the terms $m(\pi_j)$, $\dim \pi_j$, $[\mathbb{Q}(\pi_j) : \mathbb{Q}]$ are powers of 2, so gcd and minimum are the same. If p is odd, then $m(\pi_j) = 1$, and moreover, either some $\pi_j = \mathbf{1}$, in which case the claim is clear, or else all $\dim \pi_j$ are powers of p , while all $[\mathbb{Q}(\pi_j) : \mathbb{Q}]$ are $(p-1)$ times powers of p ([17, Lemma 2.1]), so again gcd and minimum are the same. \square

5. EXAMPLES: $\text{GL}_2(\mathbb{F}_q)$, $\text{PGL}_2(\mathbb{F}_q)$, $\text{SL}_2(\mathbb{F}_q)$ AND $\text{PSL}_2(\mathbb{F}_q)$

Theorem 5.1. *For every prime power $q = p^n$, the group $G = \text{GL}_2(\mathbb{F}_q)$ has $\hat{\mathbb{C}}(G) = \{1\}$.*

Proof. By Corollary 3.3, it suffices to show that every maximal quasi-elementary subgroup $Q = C \rtimes P$ of $G = \text{GL}_2(\mathbb{F}_q)$ is contained in some $\bar{Q} < G$ with $\hat{\mathbb{C}}(\bar{Q}) = 1$. Pick $C = \langle g \rangle$ cyclic, and let $P = \text{Syl}_l(N_G(C))$ for some prime number l . Write $f(t)$ for the characteristic polynomial of g .

Case 1 (split Cartan). Suppose $f(t)$ has distinct roots $a, b \in \mathbb{F}_q^\times$. Then g is conjugate to $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, and its centraliser is the split Cartan subgroup:

$$Z_G(C) \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times, \quad N_G(C) < (\mathbb{F}_q^\times \times \mathbb{F}_q^\times) \rtimes C_2,$$

with $C_2 = \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$. Here $\bar{Q} = (\mathbb{F}_q^\times \times \mathbb{F}_q^\times) \rtimes C_2$ has trivial $\hat{C}(\bar{Q})$ by Corollary 2.6.

Case 2 (non-split Cartan). Suppose $f(t)$ is irreducible over \mathbb{F}_q . Then the centraliser of C is the non-split Cartan subgroup:

$$Z_G(C) \cong \mathbb{F}_q[g]^\times \cong \mathbb{F}_{q^2}^\times, \quad N_G(C) < \mathbb{F}_{q^2}^\times \rtimes \text{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q) \cong \mathbb{F}_{q^2}^\times \rtimes C_2.$$

Again $\bar{Q} = \mathbb{F}_{q^2}^\times \rtimes C_2$ has trivial \hat{C} by Corollary 2.6.

Case 3 (scalars). Suppose $g = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ is a scalar matrix. Then $Q = C \rtimes P$ can be embedded into one of the following:

- if $l = p$: $\bar{Q} = C \times U = C \times \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$; in this case U is an elementary abelian p -group; or
- if either l is odd and $l|(q-1)$, or $l = 2$ and $q \equiv 1 \pmod{4}$: $\bar{Q} = H \rtimes C_2$ with $H = \text{split Cartan}$; or
- if either l is odd and $l|(q+1)$, or $l = 2$ and $q \equiv 3 \pmod{4}$: $\bar{Q} = H \rtimes C_2$ with $H = \text{non-split Cartan}$.

In all these cases, $\hat{C}(\bar{Q})$ is trivial by Corollary 2.6.

Case 4 (non-semisimple). Finally suppose that g is not semisimple, say $g = g_s g_u$ with g_s central and $g_u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ non-trivial unipotent. Then

$$\begin{aligned} N_G C = N_G \langle g_u \rangle &= \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid ac^{-1} \in \mathbb{F}_p^\times \right\} \\ &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{F}_q^\times \right\} \cdot \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \mid c \in \mathbb{F}_q \right\} \cdot \left\{ \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}_p^\times \right\} \\ &\cong \mathbb{F}_q^\times \times (\mathbb{F}_q \rtimes \mathbb{F}_p^\times). \end{aligned}$$

If $l = p$, then Q can be embedded into $\bar{Q} = \langle g_s \rangle \times U$, where $U = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \cong \mathbb{F}_q$ is an elementary abelian p -group. In this case $\hat{C}(\bar{Q})$ is trivial by Corollary 2.6. Otherwise, Q can be embedded into $\bar{Q} \cong (\mathbb{F}_q^\times \times \langle g_u \rangle) \rtimes \mathbb{F}_p^\times$, where the action in the semi-direct product is faithful. If τ is an irreducible character of \bar{Q} such that $\text{Res}_{\langle g_u \rangle} \tau$ is faithful, then $\bar{Q}/\ker \tau$ satisfies the assumptions of Proposition 2.7 with $K = \{1\}$, so $\text{Tr } \tau \in \text{Perm}(\bar{Q})$. Otherwise, $\text{Res}_{\langle g_u \rangle} \tau = \dim \tau \cdot \mathbf{1}$, so τ factors through an abelian quotient, and $\text{Tr } \tau \in \text{Perm}(\bar{Q})$ e.g. by Corollary 2.6. \square

Remark 5.2. It is also not hard to deduce the structure of \hat{C} for the related classical groups:

- $G = \text{PGL}_2(\mathbb{F}_q)$. Combined with Lemma 2.1, the theorem implies $\hat{C}(G) = 1$.
- $G = \text{SL}_2(\mathbb{F}_q)$. In general, $\hat{C}(G) \neq 1$. For example, $\text{SL}_2(\mathbb{F}_3)$ has $C = 1$ and $\hat{C} \cong \mathbb{Z}/2\mathbb{Z}$ (it has a 2-dimensional irreducible symplectic representation), and $\text{SL}_2(\mathbb{F}_{17})$ has $C \cong \mathbb{Z}/4\mathbb{Z}$.
- $G = \text{PSL}_2(\mathbb{F}_q)$. It is a result of Solomon, announced in [15], that $\hat{C}(G) = 1$. This can also be seen following the argument for GL_2 in Theorem 5.1: the analogues of \bar{Q} are the images of $\bar{Q} \cap \text{SL}_2(\mathbb{F}_q)$ in $\text{PSL}_2(\mathbb{F}_q)$, and they are dihedral in Cases 1 and 2 of the theorem, elementary abelian or dihedral ($p = 2$) in Case 3 and isomorphic to $\mathbb{F}_p \rtimes \mathbb{F}_p^\times$ in Case 4. Again, all these groups have $\hat{C} = 1$, so $\hat{C}(G) = 1$.

6. $\mathrm{PSL}_n(\mathbb{F}_p)$

Let ord_2 denote the 2-adic valuation of a rational number, $\mathrm{ord}_2\left(2^x \cdot \frac{a}{b}\right) = x$, where $2 \nmid ab$.

Theorem 6.1. *Let $k \geq 4$ be an integer, and p a prime. The groups $\mathrm{PSL}_k(\mathbb{F}_p)$, and therefore also $\mathrm{SL}_k(\mathbb{F}_p)$, have $\hat{C}(G)$ of exponent divisible by $2^{\min(\mathrm{ord}_2(k), \mathrm{ord}_2(p-1))}$.*

In the remainder of the section we prove the theorem using Corollary 3.10. We will construct a 2-quasi-elementary subgroup $Q = C \rtimes P$ of $G = \mathrm{PSL}_k(\mathbb{F}_p)$ and a rational character ρ of Q such that $\mathrm{Ind}_Q^G \rho$ has order divisible by $2^{\min(\mathrm{ord}_2(k), \mathrm{ord}_2(p-1))}$ in $\hat{C}(G)$.

Lemma 6.2. *Let p be an odd prime and $k \geq 4$ an integer. If $k = 4$, assume that $p \equiv 1 \pmod{4}$. Then there exists a prime number l that divides $p^{k-2} - 1$ but does not divide $p^s - 1$ for any $s < k - 2$.*

Proof. This is a special case of Zsigmondy's Theorem [18]. \square

Write Q_{2^N} for the generalised quaternion group of order 2^N .

Lemma 6.3. *The group $\mathrm{SL}_2(\mathbb{F}_q)$, $q = p^k$ has a 2-Sylow subgroup of the form*

- $S = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \cong C_p^k$ if $p = 2$;
- $S = \langle c, h \rangle \cong Q_{2^N}$, $c = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, $h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with $\alpha \in \mathbb{F}_q^\times$ of exact order $2^{N-1} || q - 1$, if $q \equiv 1 \pmod{4}$;
- $S = \langle c, h \rangle \cong Q_{2^N}$, $c = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$, $h = \begin{pmatrix} \gamma & \delta \\ \delta & -\gamma \end{pmatrix}$ with $\alpha + \beta\sqrt{-1} \in \mathbb{F}_{q^2}^\times$ of exact order $2^{N-1} || q + 1$ and any choice of $\gamma, \delta \in \mathbb{F}_q$ with $\gamma^2 + \delta^2 = -1$, if $q \equiv 3 \pmod{4}$.

Conjugation by the matrix $\iota = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is an automorphism of S , acting as -1 in the first case, as $c \mapsto c, h \mapsto h^{-1}$ in the second case, and as $c \mapsto c^{-1}, h \mapsto hc^{2m+1}$ for some m in the last case.

Proof. Direct computation. \square

From now on, G will denote $\mathrm{PSL}_k(\mathbb{F}_p)$. The theorem only has content when k is even and p is odd, so we will assume this. Write

$$n = \mathrm{ord}_2(p - 1) \geq 1, \quad N = \mathrm{ord}_2(p^{k-2} - 1) \geq 3, \quad m = \mathrm{ord}_2(k - 2) \geq 1.$$

Case A: Either $k > 4$ or $p \equiv 1 \pmod{4}$. Let A be a generator of a non-split Cartan subgroup $\mathbb{F}_{p^{k-2}}^\times = \mathrm{GL}_1(\mathbb{F}_{p^{k-2}}) < \mathrm{GL}_{k-2}(\mathbb{F}_p)$, and l a prime divisor of $p^{k-2} - 1$ as in Lemma 6.2. The conditions on l imply that the normaliser of $\langle A^{\frac{p^{k-2}-1}{l}} \rangle \cong C_l$ in $\mathrm{GL}_{k-2}(\mathbb{F}_p)$ is generated by A and by the Frobenius automorphism $F \in \mathrm{Gal}(\mathbb{F}_{p^{k-2}}/\mathbb{F}_p)$ of order $k - 2$. Note that F has determinant -1 , since it is an odd permutation on a normal basis of $\mathbb{F}_{p^{k-2}}/\mathbb{F}_p$. Define

$$\begin{aligned} c_p &= \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & I_{k-2} & \\ & & & d^{-1} \end{pmatrix}, & c_l &= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & A^{\frac{p^{k-2}-1}{l}} & \\ & & & 1 \end{pmatrix}, \\ x &= \begin{pmatrix} d^{-1} & & & \\ & 1 & & \\ & & U & \end{pmatrix}, & f &= \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & F^{(k-2)/2^m} & \end{pmatrix}, \end{aligned}$$

where $U = A^{\frac{p^k-2}{2^N}-1}$ and $d = \det U$. We view these matrices as representing elements of $G = \mathrm{PSL}_k(\mathbb{F}_p)$. Write

$$C = \langle c_p c_l \rangle \cong C_{pl}, \quad P = \langle x, f \rangle \cong C_{2^N} \rtimes C_{2^m}, \quad Q = CP \cong (C_p \times C_l) \rtimes (C_{2^N} \rtimes C_{2^m}).$$

Note that C_{2^N} acts trivially on C_l , and through a C_{2^n} quotient on C_p , while C_{2^m} acts through a C_2 quotient on C_p and faithfully on C_l .

Case B: $p \equiv 3 \pmod{4}$ and $k = 4$. We take the same c_p as in Case A, and $C = \langle c_p \rangle$. A 2-Sylow of the centraliser of C in G is isomorphic to $\{1\} \times \mathrm{Syl}_2(\mathrm{SL}_2(\mathbb{F}_p))$, which is isomorphic to Q_{2^N} by the last case of Lemma 6.3. A 2-Sylow of the normaliser is

$$P = \mathrm{Syl}_2 N_G(C) = \mathrm{Syl}_2 Z_G(C) \rtimes \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \cong Q_{2^N} \rtimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is in fact isomorphic to the semi-dihedral group $SD_{2^{N+1}}$. Again, we let $Q = CP$.

In both cases, write K for the centraliser of C in P . Thus, $K \cong C_{2^{N-n}}$ in case A, and $K \cong Q_{2^N}$ in case B, where the isomorphism is that of Lemma 6.3. Let A_p be K in case A, and a cyclic subgroup of index 2 in K that is normal in Q in case B. Let χ be faithful irreducible characters of CA_p , $\tau = \mathrm{Ind}_{CA_p}^Q \chi$ and $\rho = \mathrm{Tr} \tau \in \mathrm{Irr}_{\mathbb{Q}}(Q)$.

Lemma 6.4. *The character ρ has order 2^n in $\hat{C}(Q)$.*

Proof. We will use Proposition 2.7. The biggest subgroup of P that intersects CA_p trivially is of order 2 in case B, and of order 2^m in case A. So the order of ρ in $\hat{C}(G)$ is $2^{N+1-(N-1)-1} = 2$ in case B, and $2^{N+m-(N-n)-m} = 2^n$ in case A. \square

Finally, we show that $\mathrm{Ind}_Q^G \rho$ has order divisible by $2^{\min(\mathrm{ord}_2(k), \mathrm{ord}_2(p-1))}$ in $\hat{C}(G)$. We will use Corollary 3.10 with $Q_1 = Q_2 = Q$ and $\chi_1 = \chi_2 = \chi$. In view of Lemma 6.4, it suffices to show that

$$(6.5) \quad \sum_{[U] \in CC_{\mathrm{cyc}}(G)} S(U)$$

has 2-adic valuation at most $n - \min(\mathrm{ord}_2(k), \mathrm{ord}_2(p-1))$, where for $U \leq CA_p$,

$$S(U) = \frac{[\mathbb{Q}(\tau) : \mathbb{Q}]}{|CA_p|^2} |N_G(U)| \phi(|U|) \cdot \left(\sum_{\substack{D \leq CA_p \\ D \sim U}} \frac{\mu([D : D \cap \ker \chi])}{\phi([D : D \cap \ker \chi])} \right)^2.$$

Note that since CA_p is cyclic and χ is faithful, this simplifies to

$$S(U) = \frac{[\mathbb{Q}(\tau) : \mathbb{Q}]}{|CA_p|^2 \phi(|U|)} |N_G(U)| \mu(|U|)^2,$$

see Remark 3.11. In particular, $S(U) = 0$ if U has non-square-free order.

Case A.

The subgroups of CK of square-free order are C_{2lp} , C_{lp} , C_{2l} , C_l , C_{2p} , C_p , C_2 , and C_1 . We will show that $S(C_{lp}) + S(C_{2lp})$ has a strictly lower 2-adic valuation than the rest of the sum, and that this valuation is $n - \min(\mathrm{ord}_2(k), n)$. A summary of the calculations that follow is:

$$\begin{aligned}
\text{ord}_2[\mathbb{Q}(\tau) : \mathbb{Q}] &= \text{ord}_2(l-1) + N - n - 1 - m, \\
\text{ord}_2|CK|^2 &= 2(N-n), \\
\phi(|C_{lp}|) = \phi(|C_{2lp}|) &= (l-1)(p-1), \\
|N_G(C_{lp})| = |N_G(C_{2lp})| &= \frac{(k-2)p(p^{k-2}-1)(p-1)}{\gcd(k, p-1)}, \\
\text{ord}_2(S(C_{lp}) + S(C_{2lp})) &= \text{ord}_2(2S(C_{lp})) \\
&= 1 + \text{ord}_2(l-1) + N - n - 1 - m - 2(N-2) + N + \\
&\quad n + m - \min(\text{ord}_2(k), n) - \text{ord}_2(l-1) + n \\
&= n - \min(\text{ord}_2(k), n).
\end{aligned}$$

The assertions concerning $|CK|$ and $\phi(|C_{lp}|)$ are clear.

Since the conjugation action of P on $\text{Irr}(CK)$ is through Galois automorphisms, and $\ker(P \rightarrow \text{Aut}(CK))$ has index 2^{n+m} in P , we have

$$[\mathbb{Q}(\tau) : \mathbb{Q}] = 2^{-n-m}[\mathbb{Q}(\chi) : \mathbb{Q}] = \frac{p-1}{2^n} \frac{(l-1)2^{N-n-1}}{2^m},$$

with 2-adic valuation $\text{ord}_2(l-1) + N - n - 1 - m$.

The normaliser $N_{\text{GL}_k(p)}$ of the preimage of C_{lp} under $\text{SL} \rightarrow \text{PSL}$ consists of block diagonal matrices, with the normaliser of non-split Cartan in the lower right corner (order $(k-2)(p^{k-2}-1)$), and a Borel subgroup in the top left (order $p(p-1)^2$). The determinant is surjective on $N_{\text{GL}_k(p)}$, and $N_{\text{GL}_k(p)}$ contains $Z(\text{GL}_k(p))$, so the normaliser of C_{lp} in PSL has order $\frac{(k-2)(p^{k-2}-1)p(p-1)}{\gcd(k, p-1)}$, with 2-adic valuation $N + n + m - \min(\text{ord}_2(k), n)$. This is also the normaliser of C_{2lp} .

It remains to show that the rest of the sum in equation (6.5) has strictly greater 2-adic valuation than $\text{ord}_2(S(C_{lp}) + S(C_{2lp}))$. If $U \leq C$, then $|N_G(U)|$ and $|N_G(UC_2)|$ agree up to a power of p , $\phi(|U|) = \phi(|UC_2|)$, while $\mu(|U|) = -\mu(|UC_2|)$. It follows that the 2-adic valuation of $S(U) + S(UC_2)$ is at least 1 greater than that of $S(U)$.

Moreover, for any $U \leq C_{lp}$, the normaliser of U in G contains that of C_{lp} , while $1/\phi(|U|)$ has strictly greater 2-adic valuation than $1/\phi(|C_{lp}|)$ whenever $U \neq C_{lp}$. This establishes the claim.

Case B. The subgroups of CA_p of square-free order are C_1 , C_2 , C_p , and C_{2p} . We will show that $\text{ord}_2(\sum S(U)) = \text{ord}_2(S(C_p) + S(C_{2p})) = 0$. Again, we summarise the calculations as follows:

$$\begin{aligned}
\text{ord}_2[\mathbb{Q}(\tau) : \mathbb{Q}] &= N - 3, \\
\text{ord}_2|CA_p|^2 &= 2N - 2, \\
\phi(|C_p|) = \phi(|C_{2p}|) &= p - 1, \\
|N_G(C_p)| = p^4|N_G(C_{2p})| &= p^4 \cdot \frac{(p-1)^3 p^2 (p+1)}{2}, \\
\text{ord}_2(S(C_p) + S(C_{2p})) &= \text{ord}_2((1+p^4)S(C_{2p})) \\
&= 1 + N - 3 - 2N + 2 - 1 + N + 1 = 0.
\end{aligned}$$

The assertions concerning $|CA_p|$ and ϕ are clear.

It follows from the description of the P -action on $\text{Irr}(CK)$ that $[\mathbb{Q}(\tau) : \mathbb{Q}] = \frac{1}{2}[\mathbb{Q}(\chi) : \mathbb{Q}]$, and has 2-adic valuation 2^{N-3} .

The normaliser of C_{2p} in GL_4 is block diagonal, with all invertible matrices in the bottom right corner, and Borel in the top left. So its order in PSL is $\frac{(p-1)^3 p^2 (p+1)}{2}$ with 2-adic valuation $N + 1$. Finally, $|N(C_p)| = p^4 |N(C_{2p})|$, e.g. see Murray [10] §4.

It remains to show that the 2-adic valuation of $S(C_1) + S(C_2)$ is positive. The normaliser of C_2 in GL_4 is $\mathrm{GL}_2 \times \mathrm{GL}_2$, so the order of the normaliser in PSL is $\frac{(p-1)^3 p^2 (p+1)^2}{2}$, with 2-adic valuation $2N$, and the normaliser of C_1 is even bigger. So the 2-adic valuations of $S(C_1)$ and of $S(C_2)$ are positive.

Corollary 6.6. *As G ranges over the simple groups $\mathrm{PSL}_k(\mathbb{F}_p)$, and therefore also over $\mathrm{SL}_k(\mathbb{F}_p)$, the exponent of $C(G)_2$ is unbounded.*

Proof. If $\mathrm{ord}_2(k) > \mathrm{ord}_2(p - 1)$, then by [16, Lemma 5.6(1)] all Schur indices in $\mathrm{PSL}_k(\mathbb{F}_p)$ are trivial. So the assertion follows from Theorem 6.1. \square

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